Some inequalities related to isoperimetric inequalities with partial free boundary

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Abstract

The main purpose of this paper is to prove a sharp Sobolev inequality in the exterior of a convex bounded domain. There are two ingredients in the proof: One is the observation of some new isoperimetric inequalities with partial free boundary, and the other is an integral inequality (due to Duff [9]) for any nonnegative function under Schwarz equimeasurable rearrangement. These ingredients also allow us to establish some Moser-Trudinger type inequalities, and obtain some estimates on the principal frequency of a membrane with partial free boundary, which extend early results of Nehari [15] and Bandle [5] for two dimensional domains to the one for any dimensional domains (dimension ≥ 2).

1 Introduction

One of the main motivations of our study is to prove the following sharp Sobolev inequality.

Theorem 1.1 *Suppose that Ω is a bounded smooth convex domain in $\mathbb{R}^n$ ($n \geq 2$) and $\Omega^c = \mathbb{R}^n \setminus \overline{\Omega}$ is the exterior of Ω. For $1 \leq p < n$, we denote $p_* = np/(n-p)$, and $k(n,p) = \pi^{-1/2} n^{-1/p} (\frac{p-1}{n-p})^{1-1/p} \{\frac{\Gamma(1+n/2)\Gamma(n)}{\Gamma(n/p)\Gamma(1+n-n/p)}\}^{1/n}$ for $p > 1$ and $k(n,1) = \lim_{p \to 1+} k(n,p) = \pi^{-1/2} n^{-1} \{\Gamma(1+n/2)\}^{1/n}$ as the best constants of Sobolev inequalities on $\mathbb{R}^n$. Then*

$$(\int_{\Omega^c} |u|^{p_*})^{1/p_*} \leq 2^{1/n} k(n,p) (\int_{\Omega^c} |\nabla u|^p)^{1/p} \tag{1.1}$$

holds for all $u \in C_0^\infty(\mathbb{R}^n)$.

In the rest of the paper, we shall keep the same notations for n, p, p_* and $k(n,p)$.

The study of the sharp Sobolev inequalities goes back to Aubin and Talenti. In 1976, Aubin [2], Talenti [18] respectively, obtained the sharp Sobolev inequality on $\mathbb{R}^n$ ($n \geq 2$): Inequality

$$(\int_{\mathbb{R}^n} |u|^{p_*})^{\frac{1}{p_*}} \leq k(n,p)(\int_{\mathbb{R}^n} |\nabla u|^p)^{1/p} \tag{1.2}$$

holds for all $u(x)$ satisfying $u \in L^{p_*}(\mathbb{R}^n)$ and $\nabla u \in L^p(\mathbb{R}^n)$. For $1 < p < n$, extreme functions of (1.2) were also found by them.

Using an even reflection, we easily obtain the sharp Sobolev inequalities on the upper half space $\mathbb{R}^n_+$: If $u(x)$ satisfies $u \in L^{p_*}(\mathbb{R}^n_+)$ and $\nabla u \in L^p(\mathbb{R}^n_+)$, then

$$(\int_{\mathbb{R}^n_+} |u|^{p_*})^{\frac{1}{p_*}} \leq 2^{1/n} k(n,p)(\int_{\mathbb{R}^n} |\nabla u|^p)^{1/p}.$$

For $p = 2$ (and temporarily assuming that $n \geq 3$), noticing that the upper half space is conformally equivalent to a ball $B_r(0)$ with radius r, we can write, using the conformal invariance of the conformal Laplacian operator, the sharp Sobolev inequalities on $B_r(0)$ and on the exterior of $B_r(0)$ as follows:

$$(\int_{B_r(0)} |u|^{\frac{2n}{n-2}})^{\frac{n-2}{n}} \leq 2^{2/n} k^2(n,2)\Big(\int_{B_r(0)} |\nabla u|^2 + \frac{n-2}{2r} \int_{\partial B_r(0)} |u|^2 \Big), \qquad \forall u \in H^1(\Omega) \tag{1.3}$$

and

$$(\int_{B_r^c(0)} |u|^{\frac{2n}{n-2}})^{\frac{n-2}{n}} \leq 2^{2/n} k^2(n,2)\Big(\int_{B_r^c(0)} |\nabla u|^2 - \frac{n-2}{2r} \int_{\partial B_r(0)} |u|^2 \Big), \qquad \forall u \in C_0^\infty(\mathbb{R}^n). \tag{1.4}$$

Naturally (parallel to Aubin's conjecture which were proved by Hebey and Vaugon in [12]), one may ask if there is any sharp Sobolev inequality for any smooth bounded domain Ω. Namely, is there any constant $C(\Omega)$ so that

$$(\int_\Omega |u|^{\frac{2n}{n-2}})^{\frac{n-2}{n}} \leq 2^{2/n} k^2(n,2) \int_\Omega |\nabla u|^2 + C(\Omega) \int_{\partial\Omega} u^2 \tag{1.5}$$

holds for all $u \in H^1(\Omega)$?

Such an existence result of $C(\Omega)$ was proved by the author in a joint work with Y.Y. Li [13]. Recently there are many results concerning some similar inequalities, see for example, Hebey and Vaugon [12], Hebey [11], Y.Y.Li and M.Zhu [13]-[14], Zhu [20]-[21], Druet [8], Aubin, Druet and Hebey [3], Aubin and Li [4], and references therein. The uniform way to prove these inequalities is to argue by contradiction, which yields no information on $C(\Omega)$. So far, to my knowledge, there is no any upper bound estimate on $C(\Omega)$ unless $\partial\Omega$ is close to a sphere (see, for example, Pan and Wang [16]). From analytic point of view, it remains as a challenge to determine the optimal constant for $C(\Omega)$. Estimates on the optimal constant will shed light on

the solvability of some elliptic equations involving critical Sobolev exponents, and amazingly will have some geometric impacts (especially on isoperimetric inequalities) as we will see below.

Let us denote $C_{opt}(\Omega)$ as the optimal constant for $C(\Omega)$ in (1.5). It is well known that $C_{opt}(\Omega)$ is closely related to the geometric property of the boundary $\partial\Omega$. For instance, due to the work of Wang [19] (see also, Adimurthi and Mancini [1]), we know that for any bounded smooth domain Ω, $C_{opt}(\Omega) \geq (n-2)2^{2/n-1}k^2(n,2) \cdot \max_{x \in \partial\Omega} H(x)$, where $H(x)$ is the mean curvature function of $\partial\Omega$ with respect to the inner normal of $\partial\Omega$ (e.g., the unit sphere in $\mathbb{R}^n$ has positive mean curvature). Nevertheless, in view of (1.4), we may ask: Given a bounded convex smooth domain Ω in $\mathbb{R}^n$ (assuming $n \geq 3$), does the inequality

$$(\int_{\Omega^c} |u|^{\frac{2n}{n-2}})^{\frac{n-2}{n}} \leq 2^{2/n} k^2(n,2) \int_{\Omega^c} |\nabla u|^2$$

hold for all $u \in C_0^\infty(\mathbb{R}^n)$?

Obviously, this is a special case of the following question.

Question 1.1. Given a bounded smooth convex domain Ω in $\mathbb{R}^n$ (assuming $n \geq 2$), Does the inequality

$$(\int_{\Omega^c} |u|^{p_*})^{1/p_*} \leq 2^{1/n} k(n,p)(\int_{\Omega^c} |\nabla u|^p)^{1/p}$$

hold for all $u \in C_0^\infty(\mathbb{R}^n)$?

Our Theorem 1.1 gives an affirmative answer to this question.

One of the most interesting consequences (however, it is a *fake* consequence) of Theorem 1.1 would be the following "isoperimetric inequality with partial free boundary".

Theorem 1.2 (*isoperimetric inequality with partial free boundary*) *Let Ω be a bounded piecewise smooth domain in $\mathbb{R}^n$ ($n \geq 2$). Suppose that the boundary $\partial\Omega$ consists of two smooth hypersurfaces Γ_1 and Γ_2. If Γ_2 is concave with respect to Ω, then*

$$\frac{|\Gamma_1|}{|\Omega|^{1-1/n}} \geq \frac{\pi^{1/2}n}{\{2\Gamma(1+n/2)\}^{1/n}}, \tag{1.6}$$

where $|\Gamma_1|$ is the $n-1$-dimensional surface area of Γ_1 and $|\Omega|$ is the n-dimensional volume of Ω.

Here, we shall specify the concavity of Γ_2: We say that Γ_2 is concave with respect to Ω if any line segment between two endpoints on Γ_2 does not belong to Ω.

Theorem 1.2 is a consequence of Theorem 1.1 since it is well known that Theorem 1.1 implies Theorem 1.2 (by choosing $p = 1$ and $u(x)$ being the approximation of a characteristic function). However, we say that this is a *fake* consequence since, in this paper, we will use Theorem 1.2 to prove Theorem 1.1 via Schwarz symmetrization.

Actually this is the main idea in the proof of Theorem 1.1. It turns out that the proof of Theorem 1.2 becomes crucial.

We prove Theorem 1.2 by using some ideas similar to Steiner symmetrization. To distinguish our method, we shall call it "the method of reflection Steiner symmetrization". The details will be addressed in Section 2.

In Section 3 we discuss the procedure of equimeasurable rearrangement (Schwarz symmetrization) and reprove an integral inequality which was initially given by Duff [9]. Such an integral inequality allows us to prove Theorem 1.1 by using Theorem 1.2 in Section 4. Actually, in Section 4 we prove the following more general sharp Sobolev inequality:

Theorem 1.3 (sharp Sobolev inequality) *Let Ω be a bounded piecewise smooth domain in $\mathbb{R}^n$ ($n \geq 2$). Suppose that the boundary $\partial\Omega$ consists of one set of disjoint smooth hypersurfaces Γ_1 and another smooth hypersurface Γ_2. If Γ_2 is concave with respect to Ω, then*

$$\left(\int_\Omega |u|^{p_*}\right)^{1/p_*} \leq 2^{1/n} k(n,p) \left(\int_\Omega |\nabla u|^p\right)^{1/p} \tag{1.7}$$

holds for all u satisfying $u \in W^{1,p}(\Omega)$ and $u = 0$ on every hypersurface in Γ_1.

The crucial lemma in the proof of Theorem 1.3 is Lemma 4.1. This lemma and other ingredients in the proof of Theorem 1.3 enable us to obtain other two results.

In the remainder of Section 4, first of all, we establish some Moser-Trudinger type inequalities. Let us start with a Moser-Trudinger type inequality on a bounded smooth domain Ω in $\mathbb{R}^2$. The following inequality was proved by Cherrier [7] (see also Chang and Yang [6], more general result concerning piecewise smooth domains was established in [6]): For any $u \in C^1(\Omega)$, if $\int_\Omega |\nabla u|^2 dz \leq 1$ and $\int_\Omega u\, dz = 0$, then

$$\int_\Omega e^{2\pi u^2} dz \leq C(|\Omega|),$$

where $C(|\Omega|)$ is a constant depending only on the measure of Ω. This is similar to a "Moser-Trudinger inequality with boundary", which we shall describe below. The standard Moser-Trudinger inequality says: If $u \in C_0^1(\Omega)$ and $\int_\Omega |\nabla u|^2 dz \leq 1$, then

$$\int_\Omega e^{4\pi u^2} dz \leq C_1(|\Omega|),$$

where $C_1(|\Omega|)$ is another constant depending only on the measure of Ω. If Ω is a piecewise smooth bounded domain whose boundary consists of two curves Γ_1 and Γ_2, where Γ_2 is part of a straight line, then for any $u \in C^1(\Omega)$, $u = 0$ on Γ_1 and $\int_\Omega |\nabla u|^2 dz \leq 1$, using the even reflection with respect to Γ_2, we have

$$\int_\Omega e^{2\pi u^2} dz \leq C_2(|\Omega|)$$

for a constant $C_2(|\Omega|)$ depending only on the measure of Ω.

This leads us to ask: in the inequality (and higher dimensional analogs) due to Cherrier, Chang and Yang, for which kind of domains one can replace the condition $\int_\Omega u\,dz = 0$ by assuming that u vanishes on part of the boundary?

We give a complete answer to this question in the following theorem.

Theorem 1.4 *Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a piecewise smooth bounded domain whose boundary consists of one set of disjoint smooth hypersurfaces Γ_1 and one smooth hypersurfaces Γ_2. If Γ_2 is concave with respect to Ω, then for any $u(x)$ satisfying $u \in C^1(\Omega) \cap C(\overline{\Omega})$, $\int_\Omega |\nabla u|^n \leq 1$ and $u = 0$ on every hypersurface in Γ_1, the following inequality holds:*

$$\int_\Omega e^{\beta_n u^{\frac{n}{n-1}}}\,dz \leq C_*(|\Omega|), \tag{1.8}$$

where $C_*(|\Omega|)$ is a constant depending only on the measure of Ω, $\beta_n = n(\frac{\omega_{n-1}}{2})^{1/(n-1)}$, and ω_{n-1} is the surface area of the $(n-1)$-dimensional unit sphere.

The concavity assumption on Γ_2 is sharp. We have an example which shows that Theorem 1.4 does not hold if the concavity assumption is removed.

At the end of Section 4, we give some estimates on the principal frequency of a membrane with partial free boundary. Let $\Omega \in \mathbb{R}^2$ be a bounded piecewise smooth domain whose boundary $\partial\Omega$ containing a curve α. If α is concave with respect to Ω and Ω is simply connected, Nehari [15] proved

$$\Lambda \geq \Lambda_0,$$

where Λ is the principle frequency of a homogeneous membrane whose boundary is free along α and fixed along $\partial\Omega \setminus \alpha$, and Λ_0 is the principle frequency of a homogeneous semi-circular membrane of equal measure whose boundary is free along the diameter and fixed along the semi-circle. Later, Bandle [5] was able to remove the simply connected assumption on Ω. Their methods fail in the higher dimensional cases. Using our new isoperimetric inequalities, we are able to extend such an estimate to the one for any dimensional domains.

Theorem 1.5 *Let Ω be a bounded piecewise smooth domain in $\mathbb{R}^n$ ($n \geq 2$). Suppose that the boundary $\partial\Omega$ consists of one set of disjoint smooth hypersurfaces Γ_1 and another smooth hypersurface Γ_2. If Γ_2 is concave with respect to Ω, then*

$$\inf_{u \in H^1(\Omega)\setminus\{0\},\, u=0 \text{ on } \Gamma_1} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx} \geq \inf_{u \in H^1(B_r^+)\setminus\{0\},\, u=0 \text{ on } \partial B_r^+\setminus\{y=0\}} \frac{\int_{B_r^+} |\nabla u|^2 dx}{\int_{B_r^+} u^2 dx},$$

where B_r^+ is the upper half ball with the center at $\{0\}$ and the same volume as Ω.

2 Isoperimetric inequalities with partial free boundary

We prove Theorem 1.2 in this section.

We first observe that (1.6) holds for $n = 2$. Let Ω be a bounded piecewise smooth domain in $\mathbb{R}^2$ whose boundary consists of two smooth curves Γ_1 and Γ_2, and Γ_2 is concave with respect to Ω. We can assume that Γ_2 is not closed, otherwise the standard isoperimetric inequality yields $|\Gamma_1|/|\Omega|^{1/2} \geq 2\sqrt{\pi} > \sqrt{2\pi}$. Further, without loss of generality (due to the concavity assumption on Γ_2), we can assume that Γ_2 consists of straight line segments. If Γ_2 consists of only one straight line segment and Ω is on one side of Γ_2, by the standard isoperimetric inequality and the even reflection with respect to Γ_2, we have

$$\frac{|\Gamma_1|}{|\Omega|^{1/2}} \geq \frac{|\partial B_1^+ \cap \{(x_1, x_2) \ : \ x_2 > 0\}|}{|B_1^+|^{1/2}} = \sqrt{2\pi}.$$

Generally, we denote $\Omega = \cup_{i=1}^k D_i$ where each D_i is on one side of each extended line segment of Γ_2, and have

$$\frac{|\Gamma_1 \cap \overline{D}_i|}{|D_i|^{1/2}} \geq \sqrt{2\pi}, \quad \text{for} \quad i = 1, 2, ...k,$$

which again yields

$$\frac{|\Gamma_1|}{|\Omega|^{1/2}} \geq \frac{\sum_{i=1}^k |\Gamma_1 \cap \overline{D}_i|}{\sum_{i=1}^k |D_i|^{1/2}} \geq \sqrt{2\pi}.$$

For $n \geq 3$, if $\Gamma_1 \cap \Gamma_2$ lies in the same hyperplane, we can prove (1.6) by an even reflection just as above. In the rest of this section, we assume that $\Gamma_1 \cap \Gamma_2$ does not lie in the same hyperplane. We are going to symmetrize the domain via some even reflections. Even though the method is similar to Steiner symmetrization (see, for example, Pólya and Szegö [17]), but there are some differences in the procedures of symmetrizations. We shall call our method "the method of reflection Steiner symmetrization".

We argue by contradiction. Suppose that for some $\epsilon < \frac{1 + 2^{-1/n} - 2^{1-1/n}}{2}$,

$$\frac{|\Gamma_1|}{|\Omega|^{1-1/n}} < (2^{-1/n} - \epsilon) \cdot \frac{\pi^{1/2} n}{\{\Gamma(1 + n/2)\}^{1/n}}. \tag{2.1}$$

We are going to derive a contradiction.

We choose a suitable coordinate system so that the area of the domain on the hyperplane $\{(x_1, ..., x_n) \ : \ x_n = 0\}$ which is bounded by the projection from $\Gamma_1 \cap \Gamma_2$ to hyperplane $\{(x_1, ..., x_n) \ : \ x_n = 0\}$ is greater than zero, and $\Gamma_1 \cap \Gamma_2$ is not above (in positive x_n-direction) the tangent plane of Γ_2 which is parallel to $\{(x_1, ..., x_n) \ : \ x_n = 0\}$.

In the rest of this section, we will always denote H_θ as a hyperplane which is parallel to x_n-axis and has θ angle with x_1-axis. We need the following two lemmas.

Lemma 2.1 *If H_θ divides Ω into two equal volume domains Ω_1 and Ω_2, then at least on one of them, say Ω_1, the following inequality holds:*

$$\frac{|\Gamma_1^{1,1}|}{|\Omega_1|^{1-1/n}} < (2^{-2/n} - 2^{-1/n}\epsilon) \cdot \frac{\pi^{1/2}n}{\{\Gamma(1+n/2)\}^{1/n}},$$

where $\Gamma_1^{1,1} = \Gamma_1 \cap \overline{\Omega_1}$.

Proof. Suppose not, then

$$\frac{|\Gamma_1^{1,1}|}{|\Omega_1|^{1-1/n}} \geq (2^{-2/n} - 2^{-1/n}\epsilon) \cdot \frac{\pi^{1/2}n}{\{\Gamma(1+n/2)\}^{1/n}},$$

and

$$\frac{|\Gamma_1^{1,2}|}{|\Omega_2|^{1-1/n}} \geq (2^{-2/n} - 2^{-1/n}\epsilon) \cdot \frac{\pi^{1/2}n}{\{\Gamma(1+n/2)\}^{1/n}},$$

where $\Gamma_1^{1,2} = \Gamma_1 \cap \overline{\Omega_2}$. It follows that

$$|\Gamma_1| \geq 2(2^{-2/n} - 2^{-1/n}\epsilon) \cdot \frac{\pi^{1/2}n|\Omega_1|^{1-1/n}}{\{\Gamma(1+n/2)\}^{1/n}} = (2^{-1/n} - \epsilon) \cdot \frac{\pi^{1/2}n|\Omega|^{1-1/n}}{\{\Gamma(1+n/2)\}^{1/n}}.$$

This contradicts assumption (2.1).

Lemma 2.2 *Suppose that H_θ is a tangent plane of $\Gamma_1 \cap \Gamma_2$ and divides Ω into two domains Ω_1 and Ω_2. If $\overline{\Omega_1}$ is the domain which does not contain $\Gamma_1 \cap \Gamma_2$ completely and $|\Omega_1| \geq |\Omega_2|$, then on Ω_2 the following inequality holds:*

$$\frac{|\Gamma_1^{1,2}|}{|\Omega_2|^{1-1/n}} < (2^{-2/n} - 2^{-1/n}\epsilon) \cdot \frac{\pi^{1/2}n}{\{\Gamma(1+n/2)\}^{1/n}},$$

where $\Gamma_1^{1,2} = \Gamma_1 \cap \overline{\Omega_2}$.

Proof. Let $|\Omega_1| = r|\Omega_2|$ with $r \geq 1$. Using the even reflection of Ω_1 with respect to H_θ, we have

$$\frac{|\Gamma_1^{1,1}|}{|\Omega_1|^{1-1/n}} \geq 2^{-1/n} \cdot \frac{\pi^{1/2}n}{\{\Gamma(1+n/2)\}^{1/n}}$$

where $\Gamma_1^{1,1} = \Gamma_1 \cap \overline{\Omega_1}$. If

$$\frac{|\Gamma_1^{1,2}|}{|\Omega_2|^{1-1/n}} \geq (2^{-2/n} - 2^{-1/n}\epsilon) \cdot \frac{\pi^{1/2}n}{\{\Gamma(1+n/2)\}^{1/n}},$$

we have

$$\begin{aligned}
|\Gamma_1| &\geq (2^{-1/n}|\Omega_1|^{1-1/n} + (2^{-2/n} - 2^{-1/n}\epsilon)|\Omega_2|^{1-1/n}) \cdot \frac{\pi^{1/2}n}{\{\Gamma(1+n/2)\}^{1/n}} \\
&= (2^{-1/n}r^{1-1/n} + 2^{-2/n} - 2^{-1/n}\epsilon)|\Omega_2|^{1-1/n} \cdot \frac{\pi^{1/2}n}{\{\Gamma(1+n/2)\}^{1/n}} \\
&= 2^{-1/n}|\Omega|^{1-1/n} \cdot \frac{r^{1-1/n} + 2^{-1/n} - \epsilon}{(1+r)^{1-1/n}} \cdot \frac{\pi^{1/2}n}{\{\Gamma(1+n/2)\}^{1/n}} \\
&> 2^{-1/n}|\Omega|^{1-1/n} \cdot \frac{\pi^{1/2}n}{\{\Gamma(1+n/2)\}^{1/n}}.
\end{aligned}$$

Contradiction to (2.1).

We now start the procedure of the reflection symmetrization.

Let θ be an angle incommensurable with π. We use H_θ to cut through Ω. There are two different situations. The location of H_θ will be determined differently in these two cases.

Case 1. If there is a H_θ such that H_θ divides Ω into two equal volume domains Ω_1 and Ω_2 and $H_\theta \cap \Gamma_2 \neq \varnothing$. By Lemma 2.1, we can assume, without loss of generality, that

$$\frac{|\Gamma_1^{1,1}|}{|\Omega_1|^{1-1/n}} < (2^{-2/n} - 2^{-1/n}\epsilon) \cdot \frac{\pi^{1/2}n}{\{\Gamma(1+n/2)\}^{1/n}},$$

where $\Gamma_1^{1,1} = \Gamma_1 \cap \overline{\Omega_1}$. Similarly, we define $\Gamma_2^{1,1} = \Gamma_2 \cap \overline{\Omega_1}$. We then reflect Ω_1 with respect to H_θ and obtain Ω_1'. Let $\Omega^1 = \Omega_1 \cup \Omega_1' \cup (H_\theta \cap \Omega)$, $\Gamma_1^1 =$ the union of $\Gamma_1^{1,1}$ and its reflection, and $\Gamma_2^1 =$ the union of $\Gamma_2^{1,1}$ and its reflection. We have

$$\frac{|\Gamma_1^1|}{|\Omega^1|^{1-1/n}} < (2^{-1/n} - \epsilon) \cdot \frac{\pi^{1/2}n}{\{\Gamma(1+n/2)\}^{1/n}}. \tag{2.2}$$

Moreover, Ω^1 has the following two key properties:

Property 1. If $\{(x_1, ..., x_n) : x_n = k\} \cap \Gamma_2^1$ has a strictly concave (with respect to Ω^1) closed surface on the hyperplane $\{(x_1, ..., x_n) : x_n = k\}$, we denote M^{n-1} as the domain in the hyperplane bounded by the closed surface. Then in M^{n-1} there are points which are in a neighborhood of a boundary point on ∂M^{n-1}, and are not in Ω^1.

Property 2. The area of the projection of Γ_2^1 to $\{(x_1, ..., x_n) : x_n = 0\}$ is bounded from below by a fixed constant $C_b > 0$ and above by another fixed constant $C_a > 0$.

Property 1 follows directly from the assumption that Γ_2 is concave with respect to Ω. Since $|\Omega^1| = |\Omega|$ and the height of Ω^1 is bounded, and Ω is a bounded domain, we easily see that *Property 2* holds for Ω^1.

Case 2. If there is no hyperplane which is parallel to x_n-axis and has θ angle with x_1-axis such that it divides Ω into two equal volume domains Ω_1 and Ω_2 and has a nonempty intersection with Γ_2. By Lemma 2.2, we know that there is a domain Ω_1 which is bounded by Ω and a tangent plane H_θ of $\Gamma_1 \cap \Gamma_2$, such that $|\Omega_1| \leq |\Omega|/2$ and

$$\frac{|\Gamma_1^{1,1}|}{|\Omega_1|^{1-1/n}} < (2^{-2/n} - 2^{-1/n}\epsilon) \cdot \frac{\pi^{1/2}n}{\{\Gamma(1+n/2)\}^{1/n}},$$

where $\Gamma_1^{1,1} = \Gamma_1 \cap \overline{\Omega_1}$. Similarly, we define $\Gamma_2^{1,1} = \Gamma_2 \cap \overline{\Omega_1}$. We then reflect Ω_1 with respect to H_θ and obtain Ω_1'. Let $\Omega^1 = \Omega_1 \cup \Omega_1' \cup (H_\theta \cap \Omega)$, $\Gamma_1^1 =$ the union of $\Gamma_1^{1,1}$ and its reflection, and $\Gamma_2^1 =$ the union of $\Gamma_2^{1,1}$ and its reflection. It is easy to check that (2.2) holds for Ω^1 and Γ_1^1, and Ω^1 also has *Property 1* and *Property 2*.

We next apply $H_{2\theta}$ on Ω^1 and obtain Ω^2 with *Property 1* and *Property 2*, and so on. Eventually, we obtain a limit domain Ω^∞ with boundary Γ_1^∞ and Γ_2^∞. Due

to *Property 2*, we know that Ω^∞ will not degenerate into a straight line parallel to x_n-axis. From (2.2) we know that Ω^∞ will not degenerate into a $n-1$ domain parallel to the hyperplane $\{(x_1, ..., x_n) \ : \ x_n = 0\}$ neither. Moreover,

$$\frac{|\Gamma_1^\infty|}{|\Omega^\infty|^{1-1/n}} \leq (2^{-1/n} - \epsilon) \cdot \frac{\pi^{1/2}n}{\{\Gamma(1+n/2)\}^{1/n}}. \tag{2.3}$$

On the other hand, it is easy to see that Γ_1^∞ and Γ_2^∞ are revolution surfaces, and $\Gamma_1^\infty \cap \Gamma_2^\infty$ is a sphere lying on a hyperplane $P_\tau = \{(x_1, ..., x_n) \ : \ x_n = \tau\}$ for some constant τ. Due to *Property 1*, we know that Γ_2^∞ lies on or above the hyperplane P_τ. Let $\Omega_a^\infty = \{(x_1, ..., x_n) \in \Omega^\infty \ : \ x_n > \tau\}$, $\Omega_b^\infty = \{(x_1, ..., x_n) \in \Omega^\infty \ : \ x_n < \tau\}$, $\Gamma_{1,a}^\infty = \{(x_1, ..., x_n) \in \Gamma_1^\infty \ : \ x_n > \tau\}$, and $\Gamma_{1,b}^\infty = \{(x_1, ..., x_n) \in \Gamma_1^\infty \ : \ x_n < \tau\}$. Due to *Property 1*, using the even reflection with respect to P_τ, we have

$$|\Gamma_{1,a}^\infty| \geq 2^{-1/n} \cdot \frac{\pi^{1/2}n}{\{\Gamma(1+n/2)\}^{1/n}} \cdot |\Omega_a^\infty|^{1-1/n},$$

and

$$|\Gamma_{1,b}^\infty| \geq 2^{-1/n} \cdot \frac{\pi^{1/2}n}{\{\Gamma(1+n/2)\}^{1/n}} \cdot |\Omega_b^\infty|^{1-1/n}.$$

These two inequalities yields

$$\frac{|\Gamma_1^\infty|}{|\Omega^\infty|^{1-1/n}} \geq 2^{-1/n} \cdot \frac{\pi^{1/2}n}{\{\Gamma(1+n/2)\}^{1/n}}.$$

This contradicts (2.3)! Therefore (2.1) is false for any given $\epsilon < \frac{1+2^{-1/n}-2^{1-1/n}}{2}$. We hereby complete the proof of Theorem 1.2.

3 Equimeasurable rearrangement

Let $\Omega \subset \mathbb{R}^n$ be a bounded piecewise smooth domain and $f(x) = f(x_1, ..., x_n) \in C^1(\Omega) \cap C(\overline{\Omega})$ be a nonnegative function. In this section, we present an integral inequality for $f(x)$ under Schwarz symmetrization. This result was obtained by Duff in [9]. We present it here in a suitable way for our convenience. For completeness, we include all rigorous proofs here.

Let $\mu(t) = meas\{x \in \Omega \ : \ f(x) > t\}$. We define a one-dimensional decreasing rearrangement $f^\#(t)$ of $f(x)$ as the inverse function of $\mu(t)$, that is:

$$\mu(f^\#(t)) = t. \tag{3.1}$$

Thus the domain of $f^\#(t)$ is $[0, |\Omega|]$ and the range of $f^\#(z)$ is $[\min f(x), \max f(x)]$.

The basic relation between $df^\#(t)/dt$ and $\nabla f(x)$ is given by the following lemma.

Lemma 3.1 *For almost every value $f^{\#}(r)$ in $[\min f(x), \max f(x)]$,*

$$\frac{1}{|f^{\#'}(r)|} = \sum \int_{f=f^{\#}(r)} \frac{dS}{|\nabla f|},$$

where $f^{\#'}(r) = df^{\#}(t)/dt \mid_{t=r}$, and the integration and summation on the right hand side run over all components of the level set $f = f^{\#}(r)$ in Ω when $\{x \in \Omega : f(x) = f^{\#}(r)\}$ consists of a finite number of smooth surfaces.

Proof. From coarea formula and Sard's Theorem (see, for example, [18] or [10]), we know that for almost every $f^{\#}(r)$, $\{x \in \Omega : f(x) = f^{\#}(r)\}$ consists of a finite number of smooth surfaces, and

$$\frac{d\mu(t)}{dt} \mid_{t=f^{\#}(r)} = -\sum \int_{f=f^{\#}(r)} \frac{dS}{|\nabla f|}. \tag{3.2}$$

On the other hand, from (3.1) we have

$$\frac{d\mu(t)}{dt} \mid_{t=f^{\#}(r)} \cdot \frac{df^{\#}(r)}{dr} = 1. \tag{3.3}$$

Lemma 3.1 follows from (3.2) and (3.3) immediately.

Lemma 3.2 *If $\{x \in \Omega : f(x) = f^{\#}(r)\}$ consists of a finite number of smooth surfaces, then for any $p > 1$,*

$$\left(\sum \int_{f=f^{\#}(z)} \frac{dS}{|\nabla f|}\right)^{1-p} \leq \frac{1}{(S(f))^p} \sum \int_{f=f^{\#}(z)} |\nabla f|^{p-1} dS,$$

where $S(f) = \sum \int_{f=f^{\#}(z)} dS$.

Proof. It is equivalent to find the minimum of

$$\sum \int_{f=f^{\#}(z)} g(s)^{1-p} dS \tag{3.4}$$

for positive function $g(s)$ under the constraint:

$$\sum \int_{f=f^{\#}(z)} g(s) dS = 1.$$

Consider

$$\sum \int_{f=f^{\#}(z)} \{g(s)^{1-p} + \lambda g(s)\} dS,$$

where λ is a Lagrange multiplier. First variation of g yields the equation

$$\frac{1-p}{g(s)^p} + \lambda = 0,$$

whence
$$g(s) = 1/S(f).$$

Second variation of g is
$$p(p-1)\sum\int_{f=f^{\#}(z)}\frac{(\delta g(s))^2}{g(s)^{p+1}}dS,$$

which indicates that (3.4) attains its minimum at $g(s) = 1/S(f)$. This yields Lemma 3.2.

We are now able to establish the following key integral inequality.

Proposition 3.1 *For any $p > 1$,*
$$\int_0^{|\Omega|}|\frac{df^{\#}(z)}{dz}|^p \cdot (S(f))^p dz \le \int_\Omega |\nabla f|^p dV, \tag{3.5}$$

where $S(f) = \sum\int_{f=f^{\#}(z)}dS$ for almost every $f^{\#}(z)$.

Proof. From Lemma 3.1 and Lemma 3.2, we have
$$|\frac{df^{\#}(z)}{dz}|^{p-1} = (\sum\int_{f=f^{\#}(z)}\frac{dS}{|\nabla f|})^{1-p} \le \frac{1}{(S(f))^p}\sum\int_{f=f^{\#}(z)}|\nabla f|^{p-1}dS.$$

Thus
$$\begin{aligned}
\int_0^{|\Omega|}|\frac{df^{\#}(z)}{dz}|^p \cdot (S(f))^p dz &= \int_{z=0}^{z=|\Omega|}|\frac{df^{\#}(z)}{dz}|^{p-1} \cdot (S(f))^p|df^{\#}(z)|\\
&\le \int_{z=0}^{z=|\Omega|}\sum\int_{f=f^{\#}(z)}|\nabla f|^{p-1}dS|df^{\#}(z)|\\
&= \int_{\min f(x)}^{\max f(x)}\sum\int_{f=t}|\nabla f|^{p-1}dSdt\\
&= \int_\Omega |\nabla f|^p dV.
\end{aligned}$$

We use the coarea formula in the last equality. This completes the proof of Proposition 3.1.

Now, we introduce a radial rearrangement $f^*(x) = f^*(|x|) := f^{\#}(t)$, where
$$t = \omega_{n-1}|x|^n/n, \qquad |x|^2 = \sum_{i=1}^n x_i^2,$$

and ω_{n-1} is the surface area of the $n-1$-dimensional unit sphere. Since $dt = \omega_{n-1}|x|^{n-1}d|x|$, we have
$$|\nabla f^*(x)| = |df^{\#}(t)/dt| \cdot \omega_{n-1}|x|^{n-1} = |df^{\#}(t)/dt| \cdot S(f^*),$$

where $S(f^*) = \int_{f^*=f^{\#}(t)}dS$ is the surface area of the level surface $f^* = f^{\#}(t)$, which is a sphere in $\mathbb{R}^n$. Combining this with Proposition 3.1, we have

Corollary 3.1 *For any $p > 1$,*
$$\int_{\Omega^*}|\nabla f^*|^p \cdot \frac{(S(f))^p}{(S(f^*))^p}dV \le \int_\Omega |\nabla f|^p dV,$$

where $S(f) = \int_{f=f^*(x)}dS$ if $\{z \in \Omega : f(z) = f^*(x)\}$ consists of a finite number of smooth surfaces and $S(f^*) = \int_{f^*=f^*(x)}dS$.

4 Proofs of the theorems

Using the integral inequality in Corollary 3.1, we are able to prove Theorem 1.3, 1.4 and 1.5 from Theorem 1.2. Theorem 1.1 is simply a corollary of Theorem 1.3.

We need the following key lemma in our proofs.

Lemma 4.1 *Let Ω be a bounded piecewise smooth domain in $\mathbb{R}^n$ $(n \geq 2)$, whose boundary $\partial\Omega$ consists of one set of disjoint smooth hypersurfaces Γ_1 and another smooth hypersurface Γ_2, and Γ_2 is concave with respect to Ω. For any $u \in C^1(\Omega) \cap C(\overline{\Omega})$ with $u = 0$ on any hypersurface in Γ_1 and $u \geq 0$, we denote $u^*(x)$ as the radial rearrangement of $u(x)$ as being defined in Section 3, and Ω^* as the ball with the same volume as Ω. Then for any $p > 1$,*

$$\int_{\Omega^*} |\nabla u^*|^p dx \leq 2^{p/n} \int_{\Omega} |\nabla u|^p dx.$$

Proof. Due to Corollary 3.1 and Sard's Theorem, we only need to show that for almost every t, the level surfaces of $\{x \in \Omega : u = t\}$ and $\{x \in \Omega^* : u^*(x) = t\}$ have the following property:

$$\frac{S(u)}{S(u^*)} \geq \frac{1}{2^{1/n}}. \tag{4.1}$$

Without loss of generality, we can assume that all level surfaces $\{x \in \Omega : u = t\}$ and $\{x \in \Omega^* : u^*(x) = t\}$ are regular.

Denote $\Omega_t = \{x \in \Omega : u(x) > t\}$ and $\Omega_t^* = \{x \in \Omega^* : u^*(x) > t\}$. We know that $|\Omega_t| = |\Omega_t^*|$.

If $t = 0$, by Theorem 1.2, we have

$$\frac{S(u)}{|\Omega|^{1-1/n}} \geq \frac{|\Gamma_1|}{|\Omega|^{1-1/n}} \geq \frac{\pi^{1/2}n}{\{2\Gamma(1+n/2)\}^{1/n}}.$$

Since for any $t \geq 0$,

$$\frac{S(u^*)}{|\Omega^*_t|^{1-1/n}} = \frac{|\partial\Omega^*_t|}{|\Omega^*_t|^{1-1/n}} = \frac{\pi^{1/2}n}{\{\Gamma(1+n/2)\}^{1/n}}, \tag{4.2}$$

we have

$$\frac{S(u)}{S(u^*)} \geq \frac{1}{2^{1/n}},$$

thus (4.1) holds in this case.

If $t > 0$, we write $\partial\Omega_t \setminus \Gamma_2 = (\cup_{i=1}^k L_i) \cup (\cup_{j=1}^l \Upsilon_j)$ where L_i, Υ_j are pairwise disjoint smooth surfaces, and satisfy $\overline{L}_i \cap \Gamma_2 = \emptyset$ and $\overline{\Upsilon}_j \cap \Gamma_2 \neq \emptyset$. Since $u = 0$ on Γ_1, $\partial\Omega_t \cap \Gamma_1 = \emptyset$. Thus L_i is closed. Let Φ_i be the domain bounded by L_i and Ψ_j be the domain bounded by Υ_j and Γ_2.

From the standard isoperimetric inequality, we have

$$\frac{|L_i|}{|\Phi_i \cap \Omega|^{1-1/n}} \geq \frac{|L_i|}{|\Phi_i|^{1-1/n}} \geq \frac{|\partial B_1|}{|B_1|^{1-1/n}} = \frac{\pi^{1/2}n}{\{\Gamma(1+n/2)\}^{1/n}}, \qquad \text{for} \quad i = 1, ..., k.$$
(4.3)

On the other hand, since Γ_2 is concave we know from Theorem 1.2 that

$$\frac{|\Upsilon_j|}{|\Psi_j|^{1-1/n}} \geq \frac{\pi^{1/2}n}{\{2\Gamma(1+n/2)\}^{1/n}}, \qquad \text{for} \quad j = 1, ..., l.$$
(4.4)

It follows from (4.3) and (4.4) that

$$
\begin{aligned}
S(u) \quad &= \sum_{i=1}^{k}|L_i| + \sum_{j=i}^{l}|\Upsilon_j| \\
&\geq \frac{\pi^{1/2}n}{\{2\Gamma(1+n/2)\}^{1/n}} \cdot \left(\sum_{i=1}^{k}|\Phi_i \cap \Omega|^{1-1/n} + \sum_{j=1}^{l}|\Psi_j|^{1-1/n}\right) \\
&\geq \frac{\pi^{1/2}n}{\{2\Gamma(1+n/2)\}^{1/n}} \cdot \left(\sum_{i=1}^{k}|\Phi_i \cap \Omega| + \sum_{j=1}^{l}|\Psi_j|\right)^{1-1/n} \\
&= \frac{\pi^{1/2}n}{\{2\Gamma(1+n/2)\}^{1/n}}|\Omega_t|^{1-1/n}.
\end{aligned}
$$
(4.5)

We find on comparing (4.2) and (4.5) that (4.1) holds for $t > 0$. We hereby complete the proof of Lemma 4.1.

Proof of Theorem 1.3.

We only need to prove the sharp Sobolev inequality for $p \in (1, n)$ and for any $u \in C^1(\Omega) \cap C(\overline{\Omega})$ with $u = 0$ on any hypersurface in Γ_1 and $u \geq 0$. The sharp Sobolev inequality for $p = 1$ follows from taking $p \to 1$. We denote $u^*(x)$ as the radial rearrangement of $u(x)$ as being defined in Section 3, and Ω^* as the ball with the same volume as Ω. The equimeasurable property of the rearrangement yields

$$\int_\Omega |u|^{p*} = \int_{\Omega_*} |u^*|^{p*}.$$

Notice that $u^* = 0$ on $\partial\Omega^*$. It then follows from Lemma 4.1 and the sharp Sobolev inequality for $u^* \in W_0^{1,p}(\Omega_*)$ that

$$\frac{\int_\Omega |\nabla u|^p}{(\int_\Omega |u|^{p*})^{p/p_*}} \geq \frac{\int_{\Omega^*} |\nabla u^*|^p}{2^{p/n}(\int_{\Omega^*} |u^*|^{p*})^{p/p_*}} \geq \frac{1}{2^{p/n}k^p(n,p)}.$$

This proves inequality (1.7).

On the other hand, it is well known that

$$\inf_{v \in C^1(\Omega)\setminus\{0\}, \; v=0 \; on \; \Gamma_1} \frac{\int_\Omega |\nabla v|^p}{(\int_\Omega |v|^{p*})^{p/p_*}} \leq \frac{1}{2^{p/n}k^p(n,p)}.$$

This yields the sharpness of the constant $2^{1/n}k(n,p)$.

Proof of Theorem 1.4.

For any $u \in C^1(\Omega) \cap C(\overline{\Omega})$ with $u = 0$ on any hypersurface in Γ_1 and $u \geq 0$, we denote $u^*(x)$ as the radial rearrangement of $u(x)$ as being defined in Section 3, and Ω^* as the ball with the same volume as Ω. Notice that $u^* = 0$ on $\partial\Omega^*$.

From Lemma 4.1 we know that $||\nabla u^*||_{L^n(\Omega^*)}^n \leq 2$, thus $\int_{\Omega^*} |2^{-1/n}\nabla u^*|^n \leq 1$. It then follows from the equimeasurable property of the rearrangement and the Moser-Trudinger inequality for $u^* \in W_0^{1,n}(\Omega^*)$ that

$$\int_\Omega e^{\beta_n u^{\frac{n}{n-1}}} = \int_{\Omega^*} e^{n\omega_{n-1}^{1/(n-1)}(2^{-1/n}u^*)^{n/(n-1)}} \leq C_*(|\Omega|)$$

for some constant $C_*(|\Omega|)$ depending only on the volume of Ω. This completes the proof of Theorem 1.4.

The assumption of concavity about Γ_2 in Theorem 1.4 is crucial. Without it, Theorem 1.4 may not be true. Here is

A counterexample.

Let $\Gamma_{i,2}$ be the curve $y = a_i x^2$ for $-a_i^{-1/3} \leq x \leq a_i^{-1/3}$ and $\Gamma_{i,1}$ be another curve which connects $(-a_i^{-1/3}, a_i^{1/3})$ with $(a_i^{-1/3}, a_i^{1/3})$ and is above the straight line $y = a_i^{1/3}$ such that the area of the region Ω_i bounded by $\Gamma_{i,1}$ and $\Gamma_{i,2}$ is 1, where $a_i \to \infty$ as $i \to \infty$.

For $\lambda < e^{-a_i^2}$, we define

$$u_i = \begin{cases} \frac{\sqrt{2}}{\sqrt{2\pi}} \cdot (\ln\frac{1}{\lambda})^{1/2}, & 0 \leq |x| \leq \lambda, \\ \frac{\sqrt{2}}{\sqrt{2\pi}} \cdot \frac{\ln\frac{1}{|x|}}{(\ln\frac{1}{\lambda})^{1/2}}, & \lambda \leq |x| \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Claim:

$$\int_{\Omega_i} e^{2\pi u_i^2/||\nabla u_i||_{L^2(\Omega_i)}^2} \to \infty \qquad \text{as} \quad i \to \infty.$$

We first convert (x, y) into polar coordinates (r, θ). It is easy to see that on $\Gamma_{i,2}$ there is a $0 < \tau_0 < 1/100$ such that for $\tau_0 a_i^{-1} \leq r \leq 1$, $\theta > \tau_0/2$ in the first quadrant, and $\pi - \theta > \tau_0/2$ in the second quadrant.

Thus

$$\int_{\Omega_i} |\nabla u_i|^2 dz \leq \int_\lambda^{\tau_0 a_i^{-1}} \frac{\pi}{\pi r \ln\frac{1}{\lambda}} dr + \int_{\tau_0 a_i^{-1}}^1 \frac{\pi - \tau_0}{\pi r \ln\frac{1}{\lambda}} dr = 1 - \frac{\tau_0 \ln \tau_0^{-1} a_i}{\pi \ln\frac{1}{\lambda}}.$$

For $|x| < \lambda$,

$$\frac{2\pi u_i^2}{||\nabla u_i||_2^2} = \frac{2\ln 1/\lambda}{1 - \frac{\tau_0 \ln \tau_0^{-1} a_i}{\pi \ln\frac{1}{\lambda}}} \geq 2(1 + \frac{\tau_0 \ln \tau_0^{-1} a_i}{\pi \ln\frac{1}{\lambda}}) \cdot \ln\frac{1}{\lambda}.$$

Therefore

$$\int_{\Omega_i} e^{2\pi u_i^2/||\nabla u_i||_{L^2(\Omega_i)}^2} \geq \int_{\{(x,y)\in\Omega_i \ : \ x^2+y^2<\lambda^2\}} e^{2\pi u_i^2/||u_i||_2^2} \geq \frac{\pi\lambda^2(1+o_i(1))}{\lambda^{2+\frac{2\tau_0 \ln a_i \tau_0^{-1}}{\pi \ln 1/\lambda}}}$$

$$= \pi(1 + o_i(1))e^{\frac{2\tau_0 \ln a_i \tau_0^{-1}}{\pi}} \to \infty \qquad \text{as} \quad i \to \infty,$$

where $o_i(1) \to 0$ as $i \to \infty$ and we use $\lambda^{-1/(\ln 1/\lambda)} = e$.

Proof of Theorem 1.5.

Assume that $u \in H^1(\Omega) \setminus \{0\}$ with $u = 0$ on Γ_1. Let $u^*(x)$ be the radial rearrangement of $|u(x)|$ as being defined in Section 3, and Ω^* be the ball with the same volume as Ω. Notice that $u^* = 0$ on $\partial\Omega^*$. Using Lemma 4.1, we have:

$$\int_{\Omega^*} |\nabla u^*|^2 dx \leq 2^{2/n} \int_{\Omega} |\nabla|u||^2 dx.$$

This yields

$$\frac{\int_{\Omega} |\nabla u|^2 dx}{\int_{\Omega} u^2 dx} = \frac{\int_{\Omega} |\nabla|u||^2 dx}{\int_{\Omega} u^2 dx} \geq \frac{\int_{\Omega^*} |\nabla u^*|^2}{2^{2/n} \int_{\Omega^*} (u^*)^2 dx} \geq \frac{\lambda_1(\Omega_*)}{2^{2/n}}, \qquad (4.6)$$

where $\lambda_1(\Omega^*)$ is the first eigenvalue of $-\Delta$ with Dirichlet boundary condition in Ω^*. On the other hand, using an even reflection and a rescaling, we find that

$$\frac{\lambda_1(\Omega^*)}{2^{2/n}} = \inf_{u \in H^1(B_r^+) \setminus \{0\}, u = 0 \text{ on } \partial B_r^+ \setminus \{y=0\}} \frac{\int_{B_r^+} |\nabla u|^2 dx}{\int_{B_r^+} u^2 dx}, \qquad (4.7)$$

where B_r^+ is the upper half ball with center at $\{0\}$ and the same volume as Ω^*. From (4.6) and (4.7) we obtain Theorem 1.5.

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